

# A new injective proof of the Erdős–Ko–Rado theorem

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## Abstract

A set system  $\mathcal{F}$  is *intersecting* if for any  $F, F' \in \mathcal{F}$ ,  $F \cap F' \neq \emptyset$ . A fundamental theorem of Erdős, Ko and Rado states that if  $\mathcal{F}$  is an intersecting family of  $r$ -subsets of  $[n] = \{1, \dots, n\}$ , and  $n \geq 2r$ , then  $|\mathcal{F}| \leq \binom{n-1}{r-1}$ . Furthermore, when  $n > 2r$ , equality holds if and only if  $\mathcal{F}$  is the family of all  $r$ -subsets of  $[n]$  containing a fixed element. In this note, we provide a new injective proof of the Erdős–Ko–Rado theorem.

## 1 The Erdős–Ko–Rado theorem

For  $0 \leq j \leq n$ , let  $[j, n] = \{j, \dots, n\}$ . In particular, set  $[n] = [1, n]$ . Similarly, define  $(j, n) = \{j+1, \dots, n-1\}$ . For a set  $X$  and  $1 \leq r \leq |X|$ , denote  $2^X = \{A \mid A \subseteq X\}$  and  $\binom{X}{r} = \{A \in 2^X : |A| = r\}$ . A family  $\mathcal{F} \subseteq \binom{[n]}{r}$  is called  *$r$ -uniform*, with  $\mathcal{F}_x = \{F \in \mathcal{F} \mid x \in F\}$  called its *star centered at  $x$* . A *full star* is  $\binom{[n]}{r}_x$  for some  $x$ ; it is easy to see that  $|\binom{[n]}{r}_x| = \binom{n-1}{r-1}$ . We say that  $\mathcal{F}$  is *intersecting* if  $A \cap B \neq \emptyset$  for every  $A, B \in \mathcal{F}$ .

One of the central results in extremal set theory, the Erdős–Ko–Rado theorem finds a tight upper bound on the size of uniform intersecting set systems.

**Theorem 1.** [2] *If  $1 \leq r \leq n/2$  and  $\mathcal{F} \subseteq \binom{[n]}{r}$  is intersecting, then  $|\mathcal{F}| \leq \binom{n-1}{r-1}$ . If  $r < n/2$  then equality holds if and only if  $\mathcal{F} = \binom{[n]}{r}_x$  for some  $x \in [n]$ .*

A cornerstone of extremal combinatorics, the theorem has inspired a multitude of research avenues and applications (see [5, 6]). The original proof by Erdős, Ko and Rado made use of the now-central *shifting* technique in conjunction with an induction argument. Daykin [1] later discovered that the theorem is implied by the Kruskal–Katona theorem, while Katona [7] gave possibly the simplest proof using the notion of *cyclic permutations*. Most recently, Frankl and Füredi [4] provided another new short proof of the theorem using a non-trivial result of Katona on shadows of intersecting families.

The new proof we provide is closest in spirit to the original proof, but avoids induction and counting, and is as short as any. It relies on the shifting operation and some of its structural properties to construct an injective function that maps any intersecting family to a subfamily of  $\binom{[n]}{r}_1$ . While the

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shifting operation is injective, it is not explicitly so; that is, the shift operation on a set depends on the entire family. However, our new injection for shifted families is explicit. By direct comparison, while the approach of [4] uses an explicit complementation followed by a shadow bound, our approach uses shifting followed by an explicit complementation.

## 2 Shifting

We begin by reviewing the definition of the renowned shifting operation and state some of its important properties. For set  $A \subseteq [n]$  and  $x \in [n]$ , let  $A + x = A \cup \{x\}$ ,  $A - x = A \setminus \{x\}$ .

Define the  $(i, j)$ -shift  $\sigma_{i,j} : 2^{[n]} \rightarrow 2^{[n]}$  as follows: for  $A \in 2^{[n]}$ , let  $\sigma_{i,j}(A) = A - i + j$  if  $i \in A$  and  $j \notin A$ , and  $\sigma_{i,j}(A) = A$  otherwise. Extend this definition to  $\sigma_{i,j} : 2^{2^{[n]}} \rightarrow 2^{2^{[n]}}$  as follows: for  $\mathcal{F} \subseteq 2^{[n]}$ , let  $\sigma_{i,j}(\mathcal{F}) = \{\sigma'_{i,j}(A) \mid A \in \mathcal{F}\}$ , where  $\sigma'_{i,j}(A) = \sigma_{i,j}(A)$  if  $\sigma_{i,j}(A) \notin \mathcal{F}$ , and  $\sigma'_{i,j}(A) = A$  otherwise. The following facts are well known and easy to verify.

**Fact 2.** *For all  $A \subseteq [n]$  and all  $\mathcal{F} \subseteq 2^{[n]}$  we have*

1.  $|\sigma_{i,j}(A)| = |A|$ ,
2.  $|\sigma_{i,j}(\mathcal{F})| = |\mathcal{F}|$ , and
3. *If  $\mathcal{F}$  is intersecting then so is  $\sigma_{i,j}(\mathcal{F})$ .*

We say that a family  $\mathcal{F} \subseteq \binom{[n]}{r}$  is *shifted* if for any  $1 \leq j < i \leq n$ ,  $\sigma_{i,j}(\mathcal{F}) = \mathcal{F}$ . Frankl [3] proved the following useful proposition about shifted families.

**Proposition 3.** *Let  $\mathcal{F} \subseteq \binom{[n]}{r}$  be shifted and intersecting. Then for every  $F \in \mathcal{F}$ , there exists an  $k = k(F)$  such that  $|F \cap [2k+1]| \geq k+1$ .*

The following corollary of Proposition 3 is immediate, and will be used in the proof of Claim 5.

**Corollary 4.** *Let  $\mathcal{F} \subseteq \binom{[n]}{r}$  be shifted and intersecting, and let  $r \leq n/2$ . Then for every  $F \in \mathcal{F}$ , there exists an  $k = k(F)$  such that  $|F \cap [2k]| = k$ .*

*Proof.* Let  $F \in \mathcal{F}$  and let  $k = k(F)$  be maximum such that  $|F \cap [2k]| \geq k$ . From Proposition 3, we know that such a  $k$  exists. We claim that  $|F \cap [2k]| = k$ . If  $2k = n$ , then we have  $|F \cap [n]| = r \leq \frac{1}{2}(2k) = k$ , which implies the result, so we assume that  $2k < n$ . Suppose that  $|F \cap [2k]| \geq k+1$ . First, this implies that  $n \geq 2k+2$ . Next, the maximality of  $k$  implies that  $|F \cap [2k+2]| \leq k$ , a contradiction. Thus  $|F \cap [2k]| = k$ .  $\square$

## 3 Injection

For a shifted, intersecting family  $\mathcal{F} \subset \binom{[n]}{r}$  ( $r \leq n/2$ ), we define the function  $\phi : \mathcal{F} \rightarrow \binom{[n]}{r}_1$  as follows. For a set  $F \in \mathcal{F}$ , let  $\kappa = \kappa_F$  be maximum such that  $|F \cap [2\kappa]| = \kappa$ . We know that  $\kappa$  exists, from Corollary 4. Now, if  $1 \in F$ , let  $\phi(F) = F$ ; otherwise, let  $\phi(F) = F \triangle [2\kappa]$ . We also denote  $\phi(\mathcal{F}) = \{\phi(A) \mid A \in \mathcal{F}\}$ , as well as write  $\phi^{-1}(B) = A$  whenever  $\phi(A) = B$ , with  $\phi^{-1}(\mathcal{H}) = \{\phi^{-1}(B) \mid B \in \mathcal{H}\}$ .

**Claim 5.** For  $r \leq n/2$ , if  $\mathcal{F} \subseteq \binom{[n]}{r}$  is shifted and intersecting then the function  $\phi$  defined above is injective.

*Proof.* Let  $F_1, F_2 \in \mathcal{F}$ ,  $F_1 \neq F_2$ . If  $1 \in F_1$  and  $1 \in F_2$  then it is obvious that  $\phi(F_1) \neq \phi(F_2)$ .

Suppose that  $1 \notin F_1$  and  $1 \notin F_2$ . If  $\kappa = \kappa_{F_1} = \kappa_{F_2}$  then either  $F_1 \cap [2\kappa] \neq F_2 \cap [2\kappa]$  or  $F_1 \setminus [2\kappa] \neq F_2 \setminus [2\kappa]$ . Then the definition of  $\phi$  implies that  $\phi(F_1) \neq \phi(F_2)$ , as required. So, without loss of generality, we may assume that  $\kappa_{F_1} < \kappa_{F_2}$ . Using maximality of  $\kappa_{F_1}$ , we have that  $F_1 \setminus [2\kappa_{F_2}] \neq F_2 \setminus [2\kappa_{F_2}]$ . As  $F_1 \setminus [2\kappa_{F_2}] \subseteq \phi(F_1)$  and  $F_2 \setminus [2\kappa_{F_2}] \subseteq \phi(F_2)$ , this implies that  $\phi(F_1) \neq \phi(F_2)$ .

Finally, suppose  $1 \in F_1$  and  $1 \notin F_2$ . We need to show that  $\phi(F_2) \neq F_1$ . Suppose instead that  $\phi(F_2) = F_1$ . Let  $G_2 = F_2 \setminus [2\kappa_{F_2}]$ . Write  $G_2 = \cup_{i=0}^{p-1} [t_i, s_{i+1}]$ , where  $2\kappa_{F_2} = s_0 < t_0 \leq s_1 < t_1 \leq s_2 < \dots < t_{p-1} \leq s_p \leq t_p = n$ . For every  $h \in [0, p-1]$ , we can see that  $|\cup_{i=0}^h (s_i, t_i)| > |\cup_{i=0}^h [t_i, s_{i+1}]|$  (call this *Property  $\star$* ). Indeed, if  $|\cup_{i=0}^h (s_i, t_i)| \leq |\cup_{i=0}^h [t_i, s_{i+1}]|$ , then there exists some  $l \in [t_h, s_{h+1}]$  such that  $F_2 \cap [2l] = l$ , contradicting the maximality of  $\kappa_{F_2}$ . Let  $P \subseteq \cup_{i=0}^{p-1} (s_i, t_i)$  be the set of the smallest  $|G_2|$  elements in  $\cup_{i=0}^{p-1} (s_i, t_i)$ . It is easy to see that  $P \cap G_2 = \emptyset$ . Also, because of Property  $\star$ ,  $P$  can be obtained from  $G_2$  by a sequence of  $(i, j)$ -shifts  $\sigma_{i,j}$ . Consequently, as  $\mathcal{F}$  is shifted,  $F'_2 = F_2 \setminus G_2 \cup P \in \mathcal{F}$ . However, from the definition of  $\phi$ , and under the assumption that  $\phi(F_2) = F_1$ , we have  $G_2 \subseteq F_1$ . This implies that  $F_1 \cap F'_2 = \emptyset$ , a contradiction, as  $\mathcal{F}$  is intersecting.  $\square$

We make note of the following interesting property of the parameter  $\kappa$ . If  $\mathcal{F} \subseteq \binom{[n]}{r}$  is shifted and intersecting, with  $A \in \mathcal{F}$  and  $B = \phi(A)$ , then  $|B \cap [2k]| = |A \cap [2k]|$  for all  $k \geq \kappa_A$ . This makes it possible to define  $\kappa_B$  similarly, from which we see that  $\kappa_B = \kappa_A$ . Consequently, if we know that  $B \in \phi(\mathcal{F})$  then it must be that  $\phi^{-1}(B) \in \{B, B \Delta [2\kappa_B]\}$ . We phrase this as follows.

**Proposition 6.** For  $r \leq n/2$ , if  $\mathcal{F} \subseteq \binom{[n]}{r}$  is shifted and intersecting, with  $B \in \phi(\mathcal{F}) \setminus \mathcal{F}$ , then  $\phi^{-1}(B) = B \Delta [2\kappa_B]$ .  $\square$

## 4 Star Preservation

Here we show that the pre-shift of any full star is a full star, and also that  $\phi^{-1}(\binom{[n]}{r}_1) = \binom{[n]}{r}_1$ . Let  $G(M, s)$  be the graph on the vertex set  $\binom{M}{s}$  having edge  $AB$  (for any  $A, B \in \binom{M}{s}$ ) whenever  $|A \Delta B| = 2$ .

**Fact 7.** For  $0 \leq s \leq |M|$  we have that  $G(M, s)$  is connected.

*Proof.* The standard revolving door algorithm (Gray code for uniform subsets; see Algorithm R in Section 7.2.1.3 of [8]) shows that  $G(M, s)$  is hamiltonian.  $\square$

**Proposition 8.** For  $1 \leq r < n/2$  and intersecting  $\mathcal{F} \subset \binom{[n]}{r}$  with  $|\mathcal{F}| = \binom{n-1}{r-1}$ , let  $\mathcal{H} = \sigma_{j,i}(\mathcal{F})$  for some  $i, j \in [n]$ . Suppose that  $\mathcal{H} = \binom{[n]}{r}_i$  and define  $M = [n] \setminus \{i, j\}$ .

1. If  $A, C \in \binom{M}{r-1}$ ,  $A \cap C = \emptyset$ , and  $A + j \in \mathcal{F}$  then  $C + j \in \mathcal{F}$ .
2. If  $A, B \in \binom{M}{r-1}$ ,  $AB \in E(G(M, r-1))$ , and  $A + j \in \mathcal{F}$  then  $B + j \in \mathcal{F}$ .

*Proof.* For part (1), suppose instead that  $C + j \notin \mathcal{F}$ . Then  $C + i \in \mathcal{F}$  because  $C + i \in \mathcal{H}$ . But then  $(A + j) \cap (C + i) = \emptyset$ , a contradiction.

For part (2), given such  $A$  and  $B$ , let  $C \in \binom{M \setminus (A \cup B)}{r-1}$ , which is possible because  $|M \setminus (A \cup B)| = (n-2) - r \geq r-1$ . Then two applications of part (1) yields the result.  $\square$

**Lemma 9.** For  $1 \leq r < n/2$  and intersecting  $\mathcal{F} \subset \binom{[n]}{r}$  with  $|\mathcal{F}| = \binom{n-1}{r-1}$ , let  $\mathcal{H} = \sigma_{j,i}(\mathcal{F})$  for some  $i, j \in [n]$ . Suppose that  $\mathcal{H} = \binom{[n]}{r}_i$ . Then either  $\mathcal{F} = \binom{[n]}{r}_i$  or  $\mathcal{F} = \binom{[n]}{r}_j$ .

*Proof.* Suppose that  $\mathcal{F} \neq \binom{[n]}{r}_i$ , and define  $M = [n] \setminus \{i, j\}$ . Since  $\sigma_{j,i}(\mathcal{F}) = \mathcal{H}$ , every set in  $\mathcal{F}$  must contain either  $i$  or  $j$ . Thus there must be some  $A \subseteq M$  such that  $A + j \in \mathcal{F}$ . By Fact 7 and Proposition 8,  $\{A + j : A \in \binom{M}{r-1}\} \subseteq \mathcal{F}$ .

Also, if  $j \in S \in \mathcal{H}$  then  $S \in \mathcal{F}$ . Hence  $\mathcal{F} = \binom{[n]}{r}_j$ .  $\square$

**Lemma 10.** For  $1 \leq r < n/2$  and shifted, intersecting  $\mathcal{F} \subset \binom{[n]}{r}$  with  $\phi(\mathcal{F}) = \binom{[n]}{r}_1$ , we have  $\mathcal{F} = \binom{[n]}{r}_1$ .

*Proof.* Suppose first that  $A = \{1, n-r+2, \dots, n\} \in \mathcal{F}$ . As  $\mathcal{F}$  is shifted, this implies that  $\binom{[n]}{r}_1 \subseteq \mathcal{F}$ , as required. Thus, we may assume that  $A \notin \mathcal{F}$ . Since  $A \in \phi(\mathcal{F})$ , we have by Proposition 6 that  $\phi^{-1}(A) = \{2, n-r+2, \dots, n\} \in \mathcal{F}$ . However, because  $\mathcal{F}$  is shifted, we obtain that  $A \in \mathcal{F}$ , a contradiction.  $\square$

## 5 Proof of Theorem 1

For intersecting  $\mathcal{F} \subseteq \binom{[n]}{r}$  with  $1 \leq r \leq n/2$ , we shift  $\mathcal{F}$  until it becomes the shifted family  $\mathcal{F}'$ . Fact 2 gives  $|\mathcal{F}| = |\mathcal{F}'|$ , and Claim 5 gives  $|\mathcal{F}'| \leq \binom{n-1}{r-1}$ .

When  $r < n/2$ , Lemma 10 shows that  $\mathcal{F}'$  is a full star, and Lemma 9 shows that  $\mathcal{F}$  is a full star.

## References

- [1] D. Daykin, Erdős–Ko–Rado from Kruskal–Katona, J. Combinatorial Theory Ser. A **17** (1974) 254 – 255.
- [2] P. Erdős, C. Ko, and R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. (2) **12** (1961), 313–320.
- [3] P. Frankl, The shifting technique in extremal set theory, Surveys in combinatorics 1987 (New Cross, 1987), London Math. Soc. Lecture Note Ser., 123, Cambridge Univ. Press, Cambridge, 1987, 81 – 110.
- [4] P. Frankl, Z. Füredi, A new, short proof of the EKR theorem, J. Combinatorial Theory Ser. A **119** (2012), no. 6, 1388–1390.
- [5] C. Godsil and K. Meagher, Erdős–Ko–Rado Theorems: Algebraic Approaches, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2015.
- [6] G. Hurlbert and V. Kamat, Erdős–Ko–Rado theorems for chordal graphs and trees, J. Comb. Th. (A) **118** (2011), no. 3, 829–841.
- [7] G.O.H. Katona, A simple proof of the Erdős–Chao Ko–Rado theorem, J. Combin. Theory (B) **13** (1972), 183–184.
- [8] D.E. Knuth, The art of computer programming. Vol. 4, Fasc. 3. Generating all combinations and partitions. Addison-Wesley, Upper Saddle River, NJ, 2005.